Math Sci
DOI 10.1007/s40096-014-0136-3

ORIGINAL RESEARCH

Some inequalities associated with the Hermite–Hadamard–Fejér type for convex function

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Received: 27 August 2014 / Accepted: 19 December 2014

Abstract In this paper, we extend some estimates of the right-hand side of a Hermite–Hadamard–Fejér type inequality for functions whose first derivatives' absolute values are convex. The results presented here would provide extensions of those given in earlier works.

Keywords Hermite–Hadamard–Fejer inequality · Trapezoid inequality · Convex function · Hölder inequality.

Mathematics Subject Classification 26D07 · 26D15

Introduction

Definition 1 The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The following inequality is well known in the literature as the Hermite–Hadamard integral inequality (see, [2, 4]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

In [1], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1 Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

Theorem 1 Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}(|f'(a)| + |f'(b)|). \tag{1.3}$$

Theorem 2 Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$, $f' \in L(a, b)$ and $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}} \times \left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}. \tag{1.4}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite–Hadamard

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inequalities or its weighted versions, the so-called Hermite–Hadamard–Fejér inequalities (see [5–14]). In [3], Fejer gave a weighted generalization of the inequalities (1.1) as the following:

Theorem 3 $f : [a, b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \le \frac{1}{b-a}\int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.5}$$

holds, where $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

In [5], some inequalities of Hermite–Hadamard–Fejer type for differentiable convex mappings were proved using the following lemma.

Lemma 2 Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $f' \in L[a, b]$, then the following equality holds:

$$\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx = \frac{(b-a)^2}{2}\int_0^1 p(t)f'(ta+(1-t)b)dt \tag{1.6}$$

for each $t \in [0, 1]$, where

$$p(t) = \int_t^1 w(as+(1-s)b)ds - \int_0^t w(as+(1-s)b)ds.$$

In this article, using functions whose derivatives' absolute values are convex, we obtained new inequalities of Hermite–Hadamard–Fejer type. The results presented here would provide extensions of those given in earlier works.

Main results

We will establish some new results connected with the right-hand side of (1.5) and (1.1). Now, we prove our main theorems:

Theorem 4 Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and let $w : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|$ is convex on $[a, b]$, then for all $x \in [a, b]$, the following inequalities hold:

$$\left|\left(\int_x^b w(s)ds\right)^\alpha f(b) - \left(\int_x^a w(s)ds\right)^\alpha f(a) - \alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right|$$

$$\le \|w\|^\alpha_{[a,x],\infty}\left\{\frac{|f'(a)|}{b-a}\left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1}+\frac{(x-a)^{\alpha+2}}{\alpha+2}\right] + \frac{|f'(b)|}{b-a}\frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right\}$$

$$+\|w\|^\alpha_{[x,b],\infty}\left\{\frac{|f'(a)|}{b-a}\frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{|f'(b)|}{b-a}\left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1}+\frac{(b-x)^{\alpha+2}}{\alpha+2}\right]\right\}$$

$$\le \frac{\|w\|^\alpha_{[a,b],\infty}}{(b-a)}\left\{|f'(a)|\left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1} + \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{(x-a)^{\alpha+2}}{\alpha+2}\right]\right.$$

$$\left.+|f'(b)|\left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1}+\frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{(b-x)^{\alpha+2}}{\alpha+2}\right]\right\}$$

where $\alpha > 0$ and $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$.

Proof By integration by parts, we have the following equalities:

$$\int_a^b\left(\int_x^t w(s)ds\right)^\alpha f'(t)dt$$

$$= \left(\int_x^t w(s)ds\right)^\alpha f(t)\Big|_a^b - \alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt$$

$$= \left(\int_x^b w(s)ds\right)^\alpha f(b) - \left(\int_x^a w(s)ds\right)^\alpha f(a) - \alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt. \tag{2.1}$$

We take absolute value of (2.1) and use convexity of $|f'|$, we find that

$$\left| \left(\int_x^b w(s)ds \right)^{\alpha} f(b) - \left(\int_x^a w(s)ds \right)^{\alpha} f(a) - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t) f(t) dt \right|$$

$$\leq \int_a^x \left(\left| \int_x^t w(s)ds \right| \right)^{\alpha} |f'(t)| dt + \int_x^b \left(\left| \int_x^t w(s)ds \right| \right)^{\alpha} |f'(t)| dt$$

$$\leq \|w\|^{\alpha}_{[a,x],\infty} \int_a^x (x-t)^{\alpha} |f'(t)| dt + \|w\|_{[x,b],\infty} \int_x^b (t-x)^{\alpha} |f'(t)| dt$$

$$= \|w\|^{\alpha}_{[a,x],\infty} \left[\int_a^x (x-t)^{\alpha} \left| f'\left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b\right) \right| dt \right] + \|w\|^{\alpha}_{[x,b],\infty} \left[\int_x^b (t-x)^{\alpha} \left| f'\left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b\right) \right| dt \right]$$

$$\leq \|w\|^{\alpha}_{[a,x],\infty} \left\{ \frac{|f'(a)|}{b-a} \left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1} + \frac{(x-a)^{\alpha+2}}{\alpha+2} \right] + \frac{|f'(b)|}{b-a} \frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right\}$$
$$+ \|w\|^{\alpha}_{[x,b],\infty} \left\{ \frac{|f'(a)|}{b-a} \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{|f'(b)|}{b-a} \left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1} + \frac{(b-x)^{\alpha+2}}{\alpha+2} \right] \right\}$$

$$\leq \frac{\|w\|^{\alpha}_{[a,b],\infty}}{(b-a)} \left\{ |f'(a)| \left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1} + \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{(x-a)^{\alpha+2}}{\alpha+2} \right] + |f'(b)| \left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1} + \frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{(b-x)^{\alpha+2}}{\alpha+2} \right] \right\}$$

for all $x \in [a, b]$. Hence, the proof of theorem is completed.

Corollary 1 Under the same assumptions of Theorem 4 with $w(s) = 1$, then the following inequality holds:

$$\left| (b-x)^{\alpha} f(b) - (a-x)^{\alpha} f(a) - \alpha \int_a^b (t-x)^{\alpha-1} f(t) dt \right|$$
$$\leq \frac{1}{(b-a)} \left\{ |f'(a)| \left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1} + \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{(x-a)^{\alpha+2}}{\alpha+2} \right] + |f'(b)| \left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1} + \frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{(b-x)^{\alpha+2}}{\alpha+2} \right] \right\} \tag{2.2}$$

for all $x \in [a, b]$.

Remark 1 If we take $\alpha = 1$ and $x = \frac{a+b}{2}$ in (2.2), the inequality (2.2) reduces to (1.3).

Corollary 2 (Fejer Type Inequality) Under the same assumptions of Theorem 4 with $\alpha = 1$, then the following inequalities hold:

$$\left| f(b) \int_x^b w(s)ds + f(a) \int_a^x w(s)ds - \int_a^b w(t) f(t) dt \right|$$
$$\leq |f'(a)| \frac{(x-a)^2(3b-2a-x)\|w\|_{[a,x],\infty} + \|w\|_{[x,b],\infty}(b-x)^3}{6(b-a)} + |f'(b)| \frac{(b-x)^2(x-3a-2b)\|w\|_{[x,b],\infty} + (x-a)^3\|w\|_{[a,x],\infty}}{6(b-a)}$$
$$\leq |f'(a)| \left[\frac{(x-a)^2(3b-2a-x) + (b-x)^3}{6(b-a)} \right] \|w\|_{[a,b],\infty} + |f'(b)| \left[\frac{(b-x)^2(x-3a-2b) + (x-a)^3}{6(b-a)} \right] \|w\|_{[a,b],\infty}$$

which is proved by Tseng et al. in [8].

Corollary 3 (Weighted Trapezoid Inequality) Let $w : [a, b] \to \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x = \frac{a+b}{2}$ in Corollary 2. Then the following inequalities hold:

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(s)ds-\int_a^b w(t)f(t)dt\right|$$

$$\leq\frac{(b-a)^2}{48}\left[5\|w\|^{\alpha}_{[a,\frac{a+b}{2}],\infty}+\|w\|^{\alpha}_{[\frac{a+b}{2},b],\infty}\right]\left|f'(a)\right|$$

$$+\left[\|w\|^{\alpha}_{[a,\frac{a+b}{2}],\infty}+5\|w\|^{\alpha}_{[\frac{a+b}{2},b],\infty}\right]\left|f'(b)\right|$$

$$\leq(b-a)^2\|w\|^{\alpha}_{[a,b],\infty}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{8}\right)$$

which is proved by Tseng et al. in [8].

Theorem 5 Let $f: I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and let $w:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. If $|f'|^q$ is convex on $[a,b]$, $q>1$, then for all $x\in[a,b]$, the following inequalities hold:

$$\left|\left(\int_x^b w(s)ds\right)^{\alpha}f(b)-\left(\int_x^a w(s)ds\right)^{\alpha}f(a)\right.$$
$$\left.-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1}w(t)f(t)dt\right|$$

$$\leq\frac{(x-a)^{\alpha+\frac{1}{p}}\|w\|^{\alpha}_{[a,x],\infty}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-a)^2-(b-x)^2}{2}\left|f'(a)\right|^q+\frac{(x-a)^2}{2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\alpha+\frac{1}{p}}\|w\|^{\alpha}_{[x,b],\infty}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-x)^2}{2}\left|f'(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$\leq\frac{\|w\|^{\alpha}_{[a,b],\infty}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left\{(x-a)^{\alpha+\frac{1}{p}}\right.$$
$$\times\left(\frac{(b-a)^2-(b-x)^2}{2}\left|f'(a)\right|^q+\frac{(x-a)^2}{2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$\left.+(b-x)^{\alpha+\frac{1}{p}}\left(\left[\frac{(b-x)^2}{2}\left|f'(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f'(b)\right|^q\right]\right)^{\frac{1}{q}}\right\}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$, and $\|w\|_{\infty}=\sup_{t\in[a,b]}|w(t)|$.

Proof We take absolute value of (2.1). Using Holder's inequality, we find that

$$\left|\left(\int_x^b w(s)ds\right)^{\alpha}f(b)-\left(\int_x^a w(s)ds\right)^{\alpha}f(a)\right.$$
$$\left.-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1}w(t)f(t)dt\right|$$

$$\leq\int_a^x\left(\left|\int_x^t w(s)ds\right|\right)^{\alpha}f'(t)dt+\int_x^b\left(\left|\int_x^t w(s)ds\right|\right)^{\alpha}f'(t)dt$$

$$\leq\left(\int_a^x\left|\int_x^t w(s)ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^x\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_x^b\left|\int_x^t w(s)ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_x^b\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f'(t)\right|^q$ is convex on $[a,b]$

$$\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\leq\frac{b-t}{b-a}\left|f'(a)\right|^q+\frac{t-a}{b-a}\left|f'(b)\right|^q. \tag{2.3}$$

From (2.3), it follows that

$$\left|\left(\int_x^b w(s)ds\right)^{\alpha}f(b)-\left(\int_x^a w(s)ds\right)^{\alpha}f(a)\right.$$
$$\left.-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1}w(t)f(t)dt\right|$$

$$\leq\|w\|^{\alpha}_{[a,x],\infty}\left(\int_a^x(x-t)^{\alpha p}dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^x\left[\frac{b-t}{b-a}\left|f'(a)\right|^q+\frac{t-a}{b-a}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\|w\|^{\alpha}_{[x,b],\infty}$$
$$\times\left(\int_x^b(t-x)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_x^b\left[\frac{b-t}{b-a}\left|f'(a)\right|^q+\frac{t-a}{b-a}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$\le \frac{(x-a)^{\alpha+\frac{1}{p}}\|w\|^{\alpha}_{[a,x],\infty}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q+\frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\alpha+\frac{1}{p}}\|w\|^{\alpha}_{[x,b],\infty}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\left[\frac{(b-x)^2}{2}|f'(a)|^q+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right]\right)^{\frac{1}{q}}$$

$$\le \frac{\|w\|^{\alpha}_{[a,b],\infty}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left\{(x-a)^{\alpha+\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q\right.\right.$$

$$\left.+\frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}$$

$$\left.+(b-x)^{\alpha+\frac{1}{p}}\left(\left[\frac{(b-x)^2}{2}|f'(a)|^q+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right]\right)^{\frac{1}{q}}\right\}$$

which this completes the proof.

Corollary 4 Under the same assumptions of Theorem 5 with $w(s)=1$, then the following inequalities hold:

$$\left|(b-x)^{\alpha}f(b)-(a-x)^{\alpha}f(a)-\alpha\int_a^b(t-x)^{\alpha-1}f(t)dt\right|$$

$$\le \frac{(x-a)^{\alpha+\frac{1}{p}}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q+\frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\alpha+\frac{1}{p}}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-x)^2}{2}|f'(a)|^q+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}$$

$$\le \frac{1}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left\{(x-a)^{\alpha+\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q\right.\right.$$

$$\left.+\frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}$$

$$\left.+(b-x)^{\alpha+\frac{1}{p}}\left(\left[\frac{(b-x)^2}{2}|f'(a)|^q+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right]\right)^{\frac{1}{q}}\right\} \tag{2.4}$$

Corollary 5 Let the conditions of Corollary 4 hold. If we take $\alpha=1$ and $x=\frac{a+b}{2}$ in (2.4), then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\le \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Corollary 6 (Fejer Type Inequality) Under the same assumptions of Theorem 5 with $\alpha=1$, then the following inequalities hold:

$$\left|f(b)\int_x^b w(s)ds+f(a)\int_a^x w(s)ds-\int_a^b w(t)f(t)dt\right|$$

$$\le \frac{(x-a)^{1+\frac{1}{p}}\|w\|_{[a,x],\infty}}{(b-a)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q\right.$$

$$\left.+\frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}+\frac{(b-x)^{1+\frac{1}{p}}\|w\|_{[x,b],\infty}}{(b-a)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{(b-x)^2}{2}|f'(a)|^q\right.$$

$$\left.+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}\le\frac{\|w\|_{[a,b],\infty}}{(b-a)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left\{(x-a)^{1+\frac{1}{p}}\right.$$

$$\times\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q+\frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}$$

$$\left.+(b-x)^{1+\frac{1}{p}}\left(\left[\frac{(b-x)^2}{2}|f'(a)|^q+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right]\right)^{\frac{1}{q}}\right\}.$$

Corollary 7 (Weighted Trapezoid Inequality) Let $w:[a,b]\to\mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x=\frac{a+b}{2}$ in Corollary 6. Then the following inequalities hold:

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(s)ds-\int_a^b w(t)f(t)dt\right|$$

$$\le\frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left[\|w\|_{[a,\frac{a+b}{2}],\infty}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right.$$

$$\left.+\|w\|_{[\frac{a+b}{2},b],\infty}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\le\frac{(b-a)^2\|w\|_{[a,b],\infty}}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Theorem 6 Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and let $w : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then for all $x \in [a, b]$, the following inequality holds:

$$\left| \left(\int_x^b w(s)ds \right)^\alpha f(b) - \left(\int_x^a w(s)ds \right)^\alpha f(a) - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt \right|$$

$$\leq \frac{(b-a)^{\frac{1}{q}} \|w\|^\alpha_{[a,b],\infty}}{(\alpha p+1)^{\frac{1}{p}}} \left[(x-a)^{\alpha p+1} + (b-x)^{\alpha p+1} \right]^{\frac{1}{p}} \times \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

where $\alpha > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\|w\|_\infty = \sup_{t \in [a,b]} |w(t)|$.

Proof We take absolute value of (2.1). Using Holder's inequality and the convexity of $|f'|^q$, we find that

$$\left| \left(\int_x^b w(s)ds \right)^\alpha f(b) - \left(\int_x^a w(s)ds \right)^\alpha f(a) - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt \right|$$

$$\leq \left(\int_a^b \left| \int_x^t w(s)ds \right|^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_a^b |f'(t)|^q dt \right)^{\frac{1}{q}}$$

$$\leq \|w\|^\alpha_{[a,b],\infty} \left(\int_a^b |t-x|^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_a^b \left[\frac{b-t}{b-a} |f'(a)|^q + \frac{t-a}{b-a} |f'(b)|^q \right] dt \right)^{\frac{1}{q}} = \frac{(b-a)^{\frac{1}{q}} \|w\|^\alpha_{[a,b],\infty}}{(\alpha p+1)^{\frac{1}{p}}} \left[(x-a)^{\alpha p+1} + (b-x)^{\alpha p+1} \right]^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

which this completes the proof.

Corollary 8 Under the same assumptions of Theorem 6 with $w(s) = 1$, then the following inequality holds:

$$\left| (b-x)^\alpha f(b) - (a-x)^\alpha f(a) - \alpha \int_a^b (t-x)^{\alpha-1} f(t)dt \right|$$

$$\leq \frac{(b-a)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}} \left[(x-a)^{\alpha p+1} + (b-x)^{\alpha p+1} \right]^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}. \tag{2.5}$$

Remark 2 Let the conditions of Corollary 8 hold. If we take $\alpha = 1$ and $x = \frac{a+b}{2}$ in (2.5), then the inequality becomes the inequality (1.4).

Corollary 9 (Fejer Type Inequality) Under the same assumptions of Theorem 6 with $\alpha = 1$, then the following inequality holds:

$$\left| f(b) \int_x^b w(s)ds + f(a) \int_a^x w(s)ds - \int_a^b w(t)f(t)dt \right|$$

$$\leq \frac{(b-a)^{\frac{1}{q}} \|w\|_{[a,b],\infty}}{(p+1)^{\frac{1}{p}}} \left[(x-a)^{p+1} + (b-x)^{p+1} \right]^{\frac{1}{p}} \times \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

Corollary 10 (Weighted Trapezoid Inequality) Let $w : [a, b] \to \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x = \frac{a+b}{2}$ in Corollary 9. Then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(s)ds - \int_a^b w(t)f(t)dt \right| \leq \frac{(b-a)^2 \|w\|_{[a,b],\infty}}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

Theorem 7 Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and let $w : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then for all $x \in [a, b]$, the following inequality holds:

$$\left|\left(\int_x^b w(s)ds\right)^{\alpha} f(b)-\left(\int_x^a w(s)ds\right)^{\alpha} f(a)\right.$$
$$\left.-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right|$$
$$\leq \frac{\|w\|_{[a,b],\infty}^{\alpha}}{(\alpha+1)(\alpha+2)^{\frac{1}{q}}(b-a)^{\frac{1}{q}}}\left((x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right)^{\frac{1}{p}}$$
$$\times\left(\left((\alpha+1)(b-a)(x-a)^{\alpha+1}+(b-x)\right.\right.$$
$$\left.\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\left|f'(a)\right|^q$$
$$+\left((\alpha+1)(b-a)(b-x)^{\alpha+1}+(x-a)\right.$$
$$\left.\left.\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$, and $\|w\|_{\infty}=\sup_{t\in[a,b]}|w(t)|$.

Proof We take absolute value of (2.1). Using Holder's inequality and the convexity of $|f'|^q$, we find that

$$\left|\left(\int_x^b w(s)ds\right)^{\alpha} f(b)-\left(\int_x^a w(s)ds\right)^{\alpha} f(a)\right.$$
$$\left.-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right|$$
$$\leq\left(\int_a^b\left|\int_x^t w(s)ds\right|^{\alpha}dt\right)^{\frac{1}{p}}\left(\int_a^b\left|\int_x^t w(s)ds\right|^{\alpha}\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\|w\|_{[a,b],\infty}^{\alpha}\left(\int_a^b|t-x|^{\alpha}dt\right)^{\frac{1}{p}}\left(\int_a^b|t-x|^{\alpha}\left[\frac{b-t}{b-a}\left|f'(a)\right|^q\right.\right.$$
$$\left.\left.+\frac{t-a}{b-a}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}=\|w\|_{[a,b],\infty}^{\alpha}\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}\right)^{\frac{1}{p}}$$
$$\times\left(\left(\frac{(b-x)(x-a)^{\alpha+1}}{\alpha+1}+\frac{(x-a)^{\alpha+2}}{\alpha+2}\right)\left|f'(a)\right|^q\right.$$
$$+\left(\frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right)\left|f'(b)\right|^q+\left(\frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right)\left|f'(a)\right|^q$$
$$\left.+\left(\frac{(x-a)(b-x)^{\alpha+1}}{\alpha+1}+\frac{(b-x)^{\alpha+2}}{\alpha+2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$
$$=\frac{\|w\|_{[a,b],\infty}^{\alpha}}{(\alpha+1)(\alpha+2)^{\frac{1}{q}}(b-a)^{\frac{1}{q}}}\left((x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right)^{\frac{1}{p}}$$
$$\times\left(\left((\alpha+1)(b-a)(x-a)^{\alpha+1}+(b-x)\right.\right.$$
$$\left.\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\left|f'(a)\right|^q$$
$$+\left((\alpha+1)(b-a)(b-x)^{\alpha+1}+(x-a)\right.$$
$$\left.\left.\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

which this completes the proof.

Corollary 11 Under the same assumptions of Theorem 7 with $w(s)=1$, then the following inequality holds:

$$\left|(b-x)^{\alpha}f(b)-(a-x)^{\alpha}f(a)-\alpha\int_a^b(t-x)^{\alpha-1}f(t)dt\right|$$
$$\leq\frac{1}{(\alpha+1)(\alpha+2)^{\frac{1}{q}}(b-a)^{\frac{1}{q}}}\left((x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right)^{\frac{1}{p}}$$
$$\times\left(\left((\alpha+1)(b-a)(x-a)^{\alpha+1}+(b-x)\right.\right.$$
$$\left.\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\left|f'(a)\right|^q$$
$$+\left((\alpha+1)(b-a)(b-x)^{\alpha+1}+(x-a)\right.$$
$$\left.\left.\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}. \tag{2.6}$$

Corollary 12 Let the conditions of Corollary 11 hold. If we take $\alpha=1$ and $x=\frac{a+b}{2}$ in (2.6), then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{(b-a)}{4}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 13 (Fejer Type Inequality) Under the same assumptions of Theorem 7 with $\alpha=1$, then the following inequality holds:

$$\left|f(b)\int_x^b w(s)ds+f(a)\int_a^x w(s)ds-\int_a^b w(t)f(t)dt\right|$$
$$\leq\frac{\|w\|_{[a,b],\infty}^{\alpha}}{2\cdot 3^{\frac{1}{q}}(b-a)^{\frac{1}{q}}}\left((x-a)^2+(b-x)^2\right)^{\frac{1}{p}}$$
$$\times\left(\left(2(b-a)(x-a)^2+(b-x)\left[(x-a)^2+(b-x)^2\right]\right)\left|f'(a)\right|^q\right.$$
$$\left.+\left(2(b-a)(b-x)^2+(x-a)\left[(x-a)^2+(b-x)^2\right]\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}.$$

Corollary 14 (Weighted Trapezoid Inequality) Let $w:[a,b]\to\mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x=\frac{a+b}{2}$ in Corollary 13. Then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(s)ds-\int_a^b w(t)f(t)dt\right|$$
$$\leq\frac{(b-a)^2\|w\|_{[a,b],\infty}^{\alpha}}{4}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

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